Randomized Communication
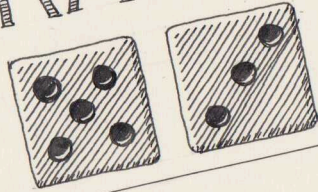
& Implicit Representations

for

Matrices & Graphs of

Small

Sign-Rank

Illustrated & Abridged!

Nathan Harms — EPFL

Viktor Zamaraev — University of Liverpool

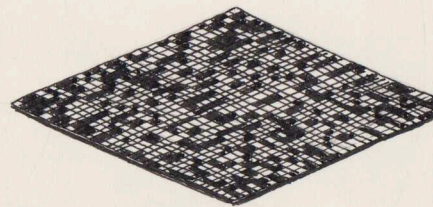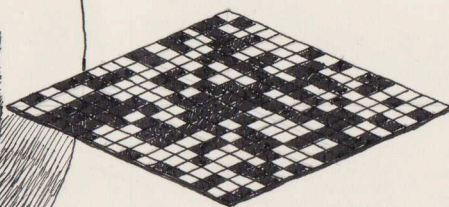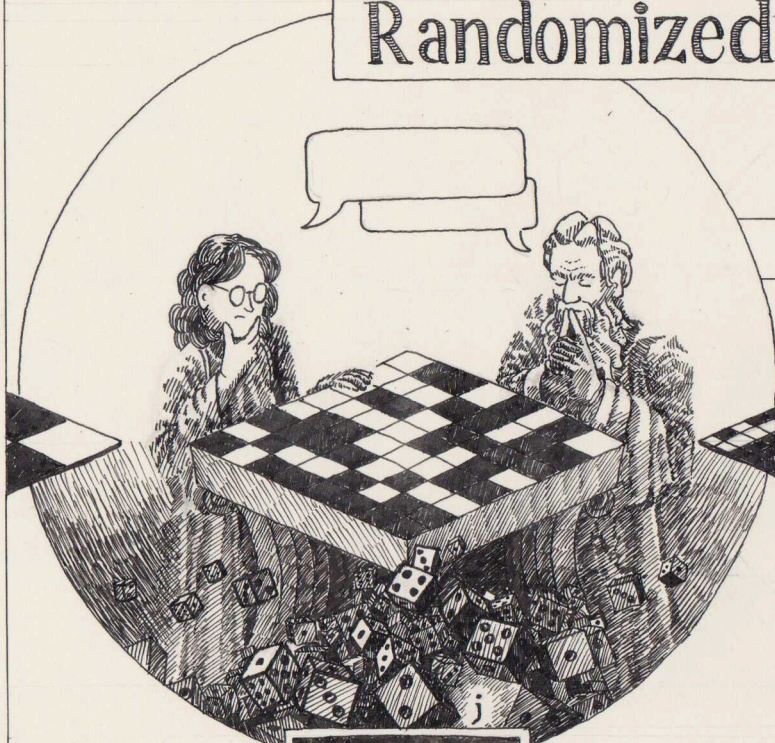Randomized
Communication
Complexity

Matrix Representations
- Bounded sign-rank
- Bounded margin
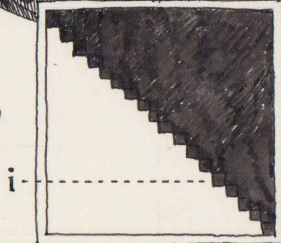- Implicit representations

THEOREM: A class of matrices of sign-rank 3 (or unit-disk graphs) has constant randomized communication cost iff it is "stable."

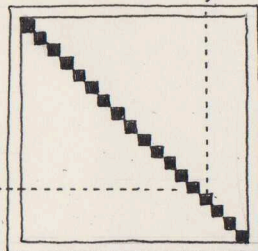# Randomized Communication

BPP⁰: Constant Cost

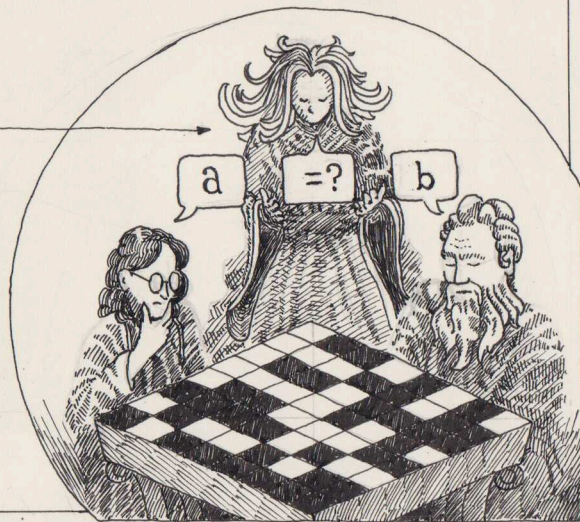GREATER-THAN
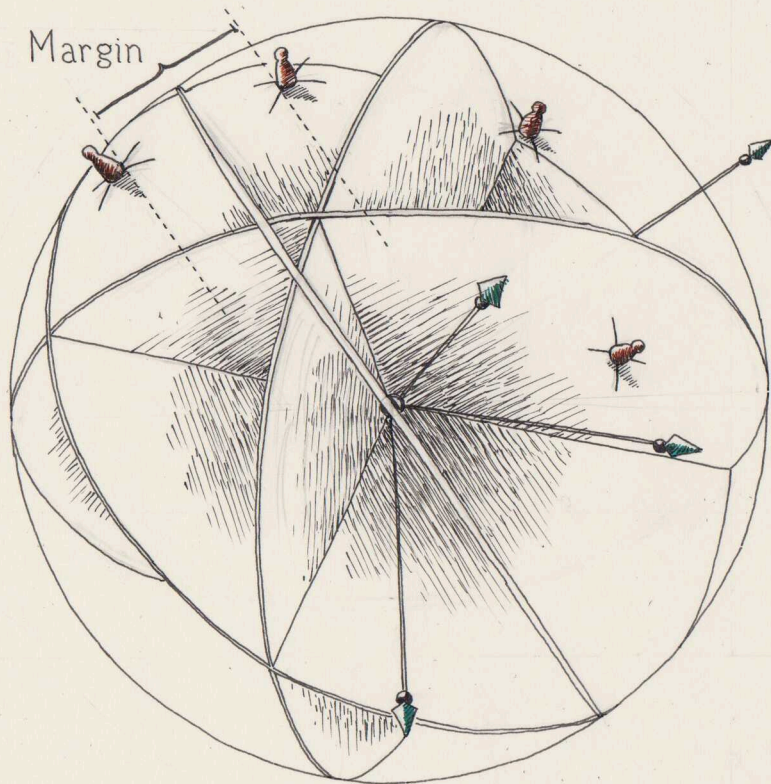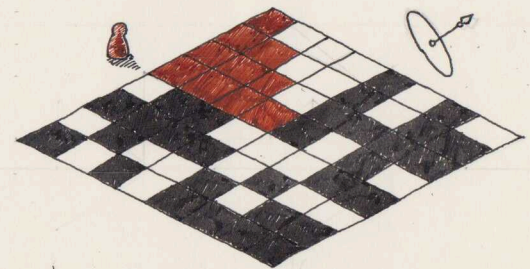
i < j?

EQUALITY

i = j?

a =? b

Bounded Margin
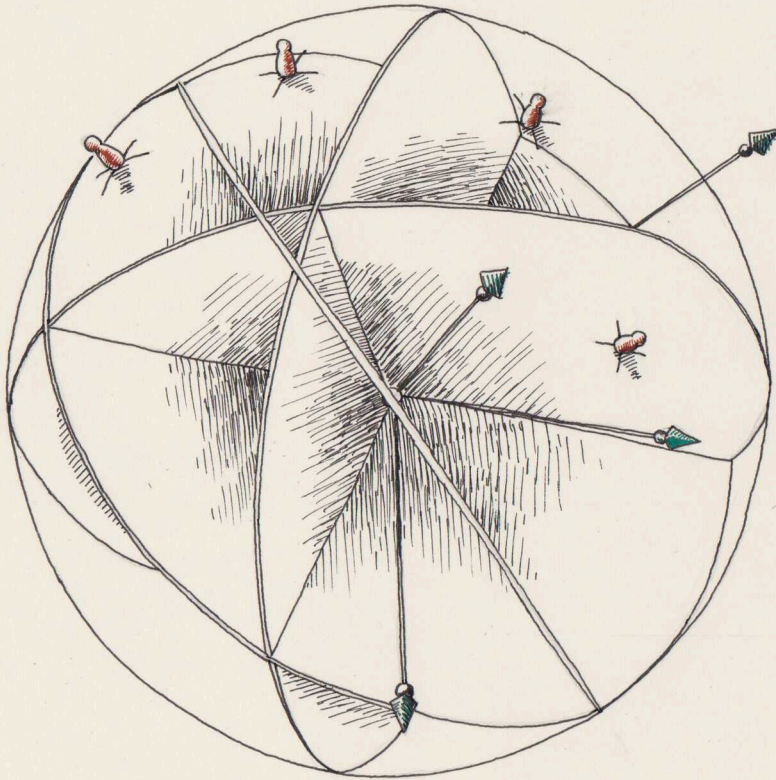

Margin

- Dimension unbounded
- Equivalent to BPP° [LS '09]
- "Stability" is necessary
    - → largest GREATER-THAN is bounded
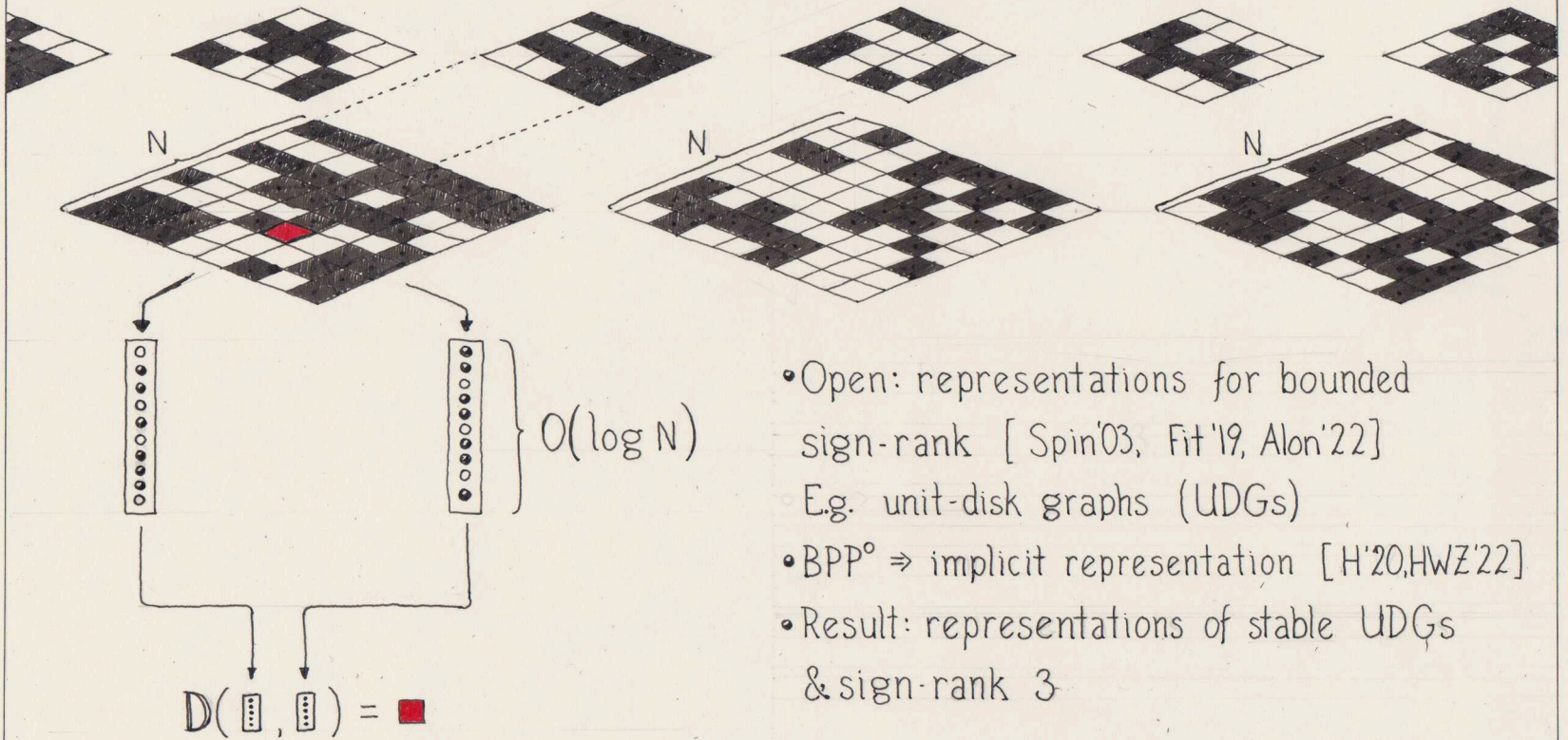- Determines performance of perceptron

Bounded Sign-Rank

- Dimension bounded
- Equivalent to UPP^0 [PS'86]
- Dimension 3 can have maximum communication cost [HHL'22, ACHS'23]
- When do 3-dimensional arrangements have constant cost?

Implicit Graph Representations

N

$O(\log N)$

$D(\square, \square) = \blacksquare$

- Open: representations for bounded sign-rank [Spin'03, Fit'19, Alon'22]
  E.g. unit-disk graphs (UDGs)
- BPP° ⇒ implicit representation [H'20, HWZ'22]
- Result: representations of stable UDGs & sign-rank 3

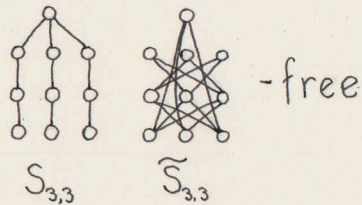**THEOREM:** Any stable set of matrices of sign-rank 3 (or UDGs) has a constant-cost deterministic EQUALITY-oracle protocol.
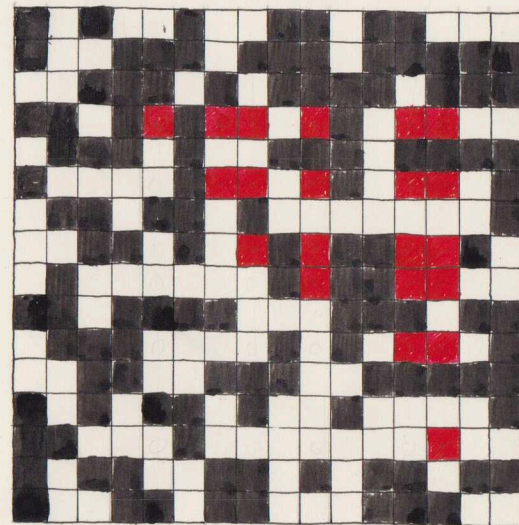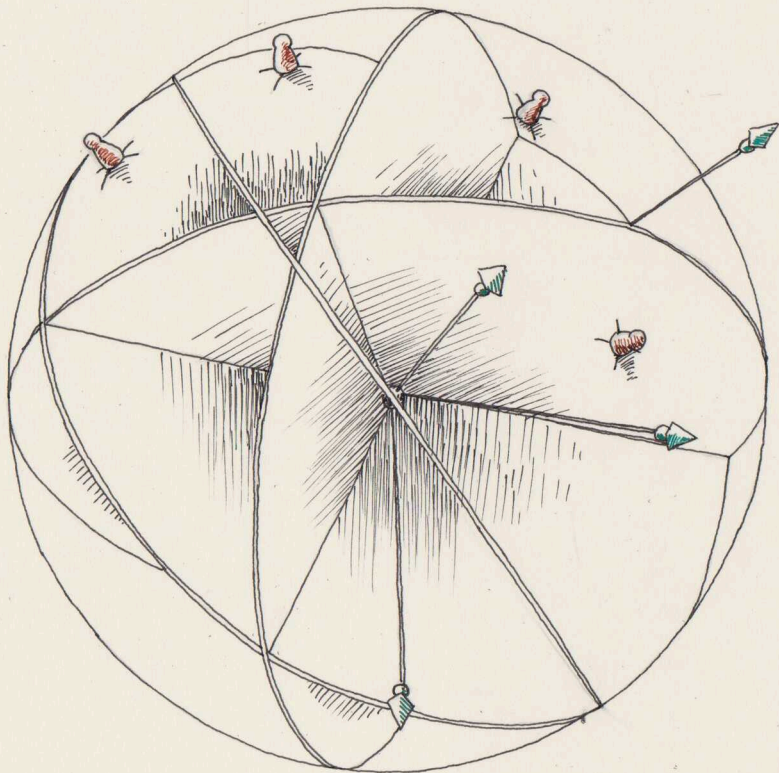
Step 1
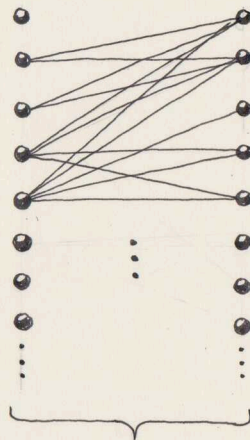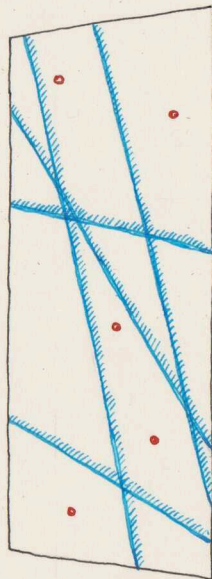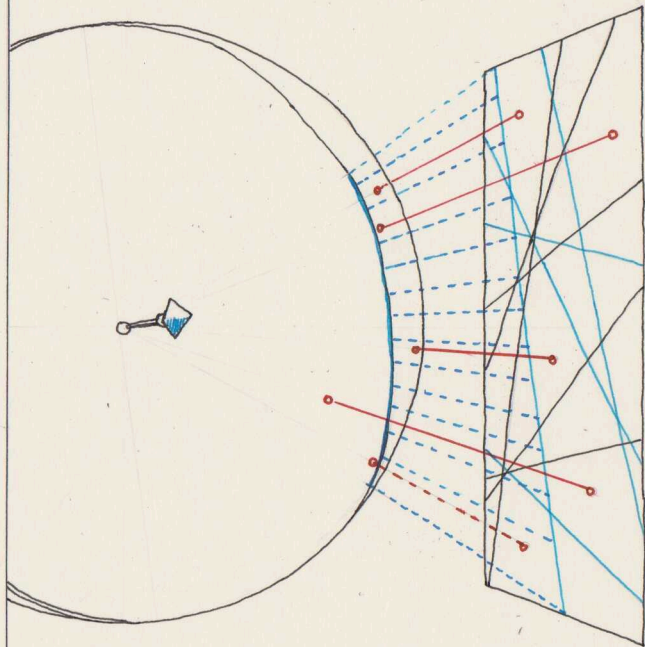
Lemma:

Constant-cost protocol for adjacency in stable $S_{s,t}, \tilde{S}_{s,t}$-free bipartite graphs.

$S_{3,3}$ $\tilde{S}_{3,3}$ -free

Step 2
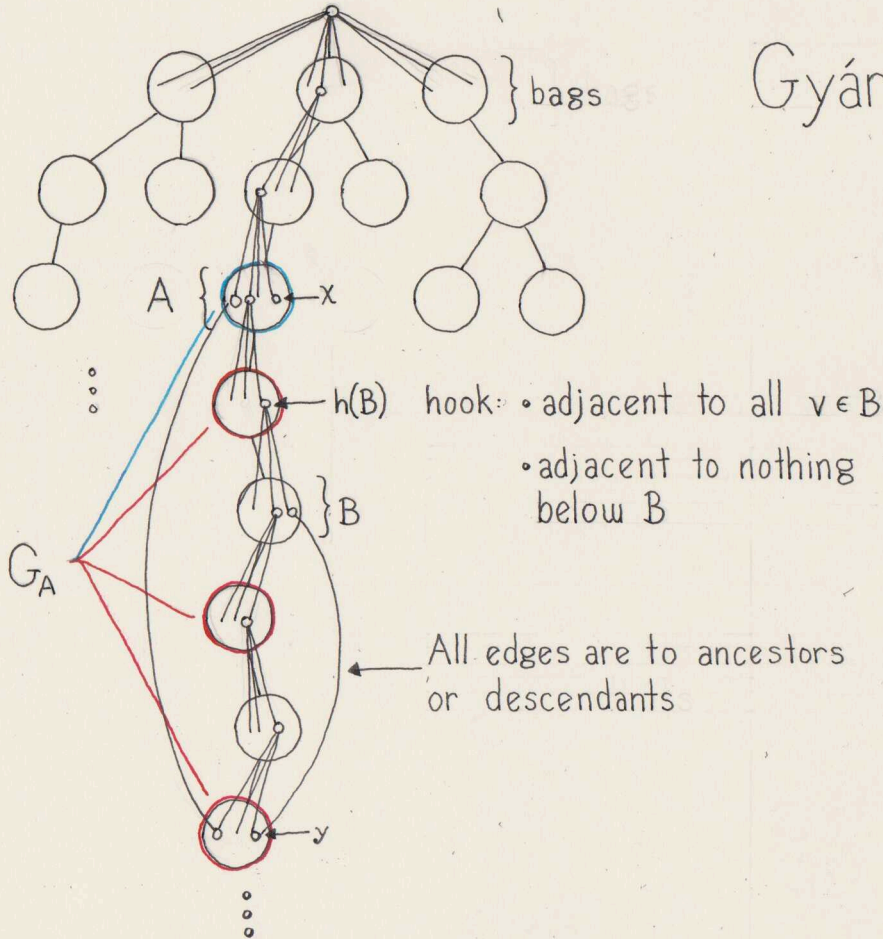
Gyárfás decomposition [POS'22]

}bags

Players agree on ancestor bag A
⇒ "Recurse on G_A or $\widetilde{G_A}$ with
  smaller GREATER-THAN."
• Stable ⇒ $O(1)$ recursions
• $S_{s,t}$, $\widetilde{S}_{s,t}$-free ⇒ always $S_{s,t}$-free

$A\{$

x

$h(B)$ hook: • adjacent to all $v \in B$
              • adjacent to nothing
                below B

$\}B$

G_A

All edges are to ancestors
or descendants

y

Step 3

Lemma: Stable, $S_{s,t}$-free $\Rightarrow$ each bag has edges to $O(1)$ ancestor bags.
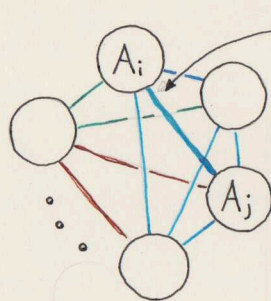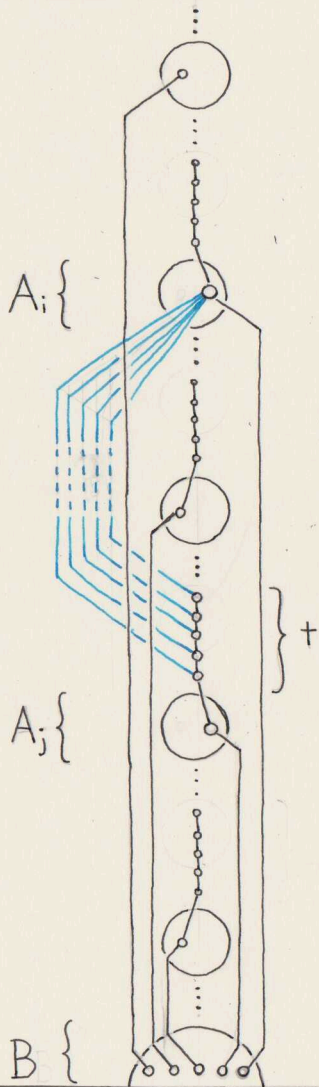
Color: induced subgraph encoding

Ramsey's theorem + case analysis on each color : find large GREATER-THAN, or $S_{s,t}$.

Players need $O(1)$ EQUALITY queries to agree on the ancestor, or output "non-adjacent."

Questions

- What is in $BPP^0 \cap UPP^0$?

  $\underbrace{\text{bounded margin}} \cap \underbrace{\text{bounded sign-rank}}$

  → Our theorem fails for sign-rank ≥ 5 [CHHS'23, CMK'19+HWZ'22+BEG]

- $BPP^0 \cap UPP^0 \overset{?}{=} (P^{EQ})^0$? ($O(1)$ EQUALITY queries)

  $\supseteq$ is known [CHHS'23]

  → Implies conjectures of [HHPTZ'22, CHHS'23]

- Implicit representations without stability, & sign-rank > 3 ?

Terminus